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ON THE RECIPROCAL DEGENERATE LAH-BELL POLYNOMIALS AND NUMBERS

BYUNG MOON KIM[1], YUNJAE KIM[2], AND JIN-WOO PARK[3]

Abstract. In this paper, we consider reciprocal degenerate Lah-Bell polynomials and numbers and study some properties for those polynomials and numbers including their explicit expressions. In addition, we find some relationships between the reciprocal Lah-Bell numbers and special numbers and polynomials.

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1. Introduction

The *unsigned Lah number* $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets and have and explicit formula

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!},\ \text{(see [2, 4, 5, 7-10])}. \tag{1}$$

By (1), we can derive the generating function of $L(n,k)$ to be

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\ (k\geq 0),\ \text{(see [2, 4, 5, 9, 10])}. \tag{2}$$

The *Bell polynomials* are defined by the generating function to be

$$e^{x\left(e^t-1\right)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\ \text{(see [2, 4, 5, 7, 9, 11, 13, 15, 16])}.$$

In the special case $x=1$, $Bel_n=Bel_n(1)$ are called the *n-th Bell numbers*. The n-th Bell number is the number of ways to partition a set with n elements into non-empty subsets. Note that

$$Bel_n=\sum_{k=0}^{n}S_2(n,k),$$

where $S_2(n,k)$ is the Stirling numbers of the second kind which is the number of ways to partition a set n elements into k non-empty subsets.

In [7], Kim and Kim introduced the *Lah-Bell polynomials* by using generating function to be

$$e^{x\left(\frac{t}{1-t}\right)}=\sum_{n=0}^{\infty}B_n^L(x)\frac{t^n}{n!}. \tag{3}$$

When $x = 1$, $B_n^L = B_n^L(1)$ are called the *n-th Lah-Bell numbers*. Note that B_n^L is the number of ways a set of n elements can be partitioned into non-empty linearly ordered subsets.

For each $\lambda \in \mathbb{R} - \{0\}$, the *degenerate exponential function* is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \text{ and } e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{4}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$ and $\lim_{\lambda\to 0} e_\lambda(t) = e^t$.

By using (4), the *degenerate Lah-Bell polynomials* are defined by the generating function to be

$$e_\lambda^x\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!}, \text{ (see [4, 5]).} \tag{5}$$

In the special case $x = 1$, $B_{n,\lambda}^L = B_{n,\lambda}^L(1)$ are called *degenerate Lah-Bell numbers*.

The study of degenerate versions of Bernoulli and Euler polynomials and numbers is initiated by Carlitz in [1], and recently, many researcher investigated various degenerate version of some special functions (see [3, 5, 6, 12, 15-18]). These studies are important not only in number theory and combinatorics but also in probability theory and applications of differential equations.

In this paper, we consider reciprocal degenerate Lah-Bell polynomials and numbers and study some properties and identities for those polynomials and numbers. In addition, we find some relationships between the reciprocal Lah-Bell numbers and special numbers and polynomials.

2. Reciprocal degenerate Lah-Bell polynomials

First of all, we observe that $e_\lambda^{-x}\left(\frac{t}{1-t}\right)$ is the reciprocal of the generating function of degenerate Lah-Bell polynomials $B_{n,\lambda}^L(x)$. We define the *reciprocal degenerate Lah-Bell polynomials* $\mathcal{B}_{n,\lambda}(x)$ whose generating function to be

$$e_\lambda^{-x}\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{6}$$

In the special case $x = 1$, $\mathcal{B}_{n,\lambda} = \mathcal{B}_{n,\lambda}(1)$ are called the *reciprocal Lah-Bell numbers*.

Note that

$$\begin{aligned} \frac{1}{(1-t)^m} &= \sum_{l=0}^{\infty} \binom{-m}{l}(-t)^l \\ &= \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \end{aligned} \tag{7}$$

where $< x >_0 = 1$, $< x >_n = x(x+1)(x+2)\cdots(x+(n-1))$ is the rising factorial of x. In addition, $< x >_{0,\lambda} = 1$, $< x >_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

Theorem 2.1. *For each nonnegative integer* n*, we have*

$$\begin{aligned}\mathcal{B}_{n,\lambda}(x) &= \sum_{m=0}^{n}(-1)^m L(n,m) < x >_{m,\lambda} \\ &= \sum_{m=0}^{n}\binom{n}{m}(-1)^m < m >_{n-m} < x >_{m,\lambda} .\end{aligned}$$

In particular,

$$\begin{aligned}\mathcal{B}_{n,\lambda} &= \sum_{m=0}^{n}(-1)^m L(n,m) < 1 >_{m,\lambda} \\ &= \sum_{m=0}^{n}\binom{n}{m}(-1)^m < m >_{n-m} < 1 >_{m,\lambda} .\end{aligned}$$

Proof. By (2) and (6), we get

$$\begin{aligned}e_\lambda^{-x}\left(\frac{t}{1-t}\right) &= \left(1+\lambda\left(\frac{t}{1-t}\right)\right)^{-\frac{x}{\lambda}} = \sum_{m=0}^{\infty}\binom{-\frac{x}{\lambda}}{m}\lambda^m\left(\frac{t}{1-t}\right)^m \\ &= \sum_{m=0}^{\infty}(-1)^m < x >_{m,\lambda} \frac{1}{m!}\left(\frac{t}{1-t}\right)^m \\ &= \sum_{m=0}^{\infty}(-1)^m < x >_{m,\lambda} \sum_{l=m}^{\infty} L(l,m)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^m L(n,m) < x >_{m,\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

On the other hand, by (7), we have

$$\begin{aligned}e_\lambda^{-x}\left(\frac{t}{1-t}\right) &= \sum_{m=0}^{\infty}(-1)^m < x >_{m,\lambda} \frac{1}{m!}\left(\frac{t}{1-t}\right)^m \\ &= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(-1)^m < x >_{m,\lambda} < m >_l \frac{t^{m+l}}{m!l!} \qquad (8) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(-1)^m < m >_{n-m} < x >_{m,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

and thus our proofs are completed. □

Note that

$$
\begin{aligned}
&\left(1+\lambda\frac{t}{1-t}\right)^{-1}\frac{1}{(1-t)^2}\\
&=\sum_{l=0}^{\infty}\binom{-1}{l}\lambda^l\left(\frac{t}{1-t}\right)^l\frac{1}{(1-t)^2}=\sum_{l=0}^{\infty}\frac{(-1)^l<1>_l}{l!}\lambda^l t^l(1-t)^{-l-2}\\
&=\left(\sum_{l=0}^{\infty}\frac{(-1)^l<1>_l}{l!}\lambda^l t^l\right)\left(\sum_{m=0}^{\infty}<l+2>_m\frac{t^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l<1>_l<l+2>_{n-l}\right)\frac{t^n}{n!}.
\end{aligned}\tag{9}
$$

Theorem 2.2. *For each nonnegative integer n, we have*

$$
\begin{aligned}
\mathcal{B}_{n+1,\lambda}(x)&=-x\sum_{m=0}^{n}\binom{n}{m}<2>_{n-m}\mathcal{B}_{m,\lambda}(x+\lambda)\\
&=-x\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-\lambda)^l l!<l+1>_{k-l}\mathcal{B}_{n-k}(x).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\mathcal{B}_{n+1,\lambda}&=-\sum_{m=0}^{n}\binom{n}{m}<2>_{n-m}\mathcal{B}_{m,\lambda}(1+\lambda)\\
&=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^{l+1}\lambda^l l!<l+1>_{k-l}\mathcal{B}_{n-k}.
\end{aligned}
$$

Proof. By taking the derivative with respect to t on both sides of (6), we get

$$
\frac{d}{dt}\left(\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\mathcal{B}_{n+1,\lambda}(x)\frac{t^n}{n!},\tag{10}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\left(e_{\lambda}^{-x}\left(\frac{t}{1-t}\right)\right)=\frac{d}{dt}\left(1+\lambda\frac{t}{1-t}\right)^{-\frac{x}{\lambda}}\\
&=-x\left(1+\lambda\frac{t}{1-t}\right)^{-\frac{x+\lambda}{\lambda}}(1-t)^{-2}\\
&=-x\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}<2>_{n-m}\mathcal{B}_{m,\lambda}(x+\lambda)\right)\frac{t^n}{n!}.
\end{aligned}\tag{11}
$$

By (11) and (10), we have

$$
\mathcal{B}_{n+1,\lambda}(x)=-x\sum_{m=0}^{n}\binom{n}{m}<2>_{n-m}\mathcal{B}_{m,\lambda}(x+\lambda).
$$

On the other hand, by (9), we get

(12)
$$\begin{aligned}&\frac{d}{dt}\left(e_\lambda^{-x}\left(\frac{t}{1-t}\right)\right)=\frac{d}{dt}\left(1+\lambda\frac{t}{1-t}\right)^{-\frac{x}{\lambda}}\\&=-x\left(1+\lambda\frac{t}{1-t}\right)^{-\frac{x}{\lambda}}\left(1+\lambda\frac{t}{1-t}\right)^{-1}(1-t)^{-2}\\&=-x\left(\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l<1>_l<l+2>_{n-l}\right)\frac{t^n}{n!}\right)\\&=-x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-\lambda)^l<1>_l<l+2>_{k-l}\mathcal{B}_{n-k,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned}$$

By (10), (11) and (12), we obtain

$$\mathcal{B}_{n+1,\lambda}(x)=-x\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-\lambda)^l<1>_l<l+2>_{k-l}\mathcal{B}_{n-k,\lambda}(x).$$

□

From now on, we find the relationship between degenerate Lah-Bell polynomials and reciprocal degenerate Lah-Bell polynomials.

Theorem 2.3. *For each nonnegative integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}B^L_{n-l,\lambda}(x)\mathcal{B}_{l,\lambda}(x)=\delta_{0,n},$$

where $\delta_{0,\lambda}$ is the Kronecker's symbol.

In particular,

$$\sum_{l=0}^{n}\binom{n}{l}B^L_{n-1,\lambda}\mathcal{B}_{l,\lambda}=\delta_{0,n}.$$

Proof. By (5) and (6), we get

$$\begin{aligned}1&=e_\lambda^{x}\left(\frac{t}{1-t}\right)e_\lambda^{-x}\left(\frac{t}{1-t}\right)\\&=\left(\sum_{m=0}^{\infty}B^L_{m,\lambda}(x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\mathcal{B}_{l,\lambda}(x)\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B^L_{n-l,\lambda}(x)\mathcal{B}_{l,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned}\tag{13}$$

By (13), we proof is completed. □

Note that, by taking the derivative with respect to t on both sides of (5), we get

$$\begin{aligned}\frac{d}{dt}\left(e_\lambda^x\left(\frac{t}{1-t}\right)\right) =&\frac{d}{dt}\left(1+\lambda\frac{t}{1-t}\right)^{\frac{x}{\lambda}}\\ =&x\left(1+\lambda\frac{t}{1-t}\right)^{\frac{x}{\lambda}-1}\frac{1}{(1-t)^2}\\ =&x\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}<2>_{n-m}B_{m,\lambda}^{L}(x-\lambda)\right)\frac{t^n}{n!}.\end{aligned}\tag{14}$$

In addition, by (9) and (14), we have

$$\begin{aligned}&\frac{d}{dt}\left(e_\lambda^x\left(\frac{t}{1-t}\right)\right)\\ =&x\left(1+\lambda\frac{t}{1-t}\right)^{\frac{x}{\lambda}}\frac{1}{(1-t)^2}\\ =&x\left(1+\lambda\frac{t}{1-t}\right)^{\frac{x}{\lambda}}\left(1+\lambda\frac{t}{1-t}\right)^{-1}(1-t)^{-2}\\ =&x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-\lambda)^l<1>_l<l+2>_{k-l}B_{n-k,\lambda}^{L}(x)\right)\frac{t^n}{n!}.\end{aligned}\tag{15}$$

Since

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty}B_{n,\lambda}^{L}(x)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}B_{n+1,\lambda}^{L}(x)\frac{t^n}{n!},\tag{16}$$

by (14), (15) and (16), we obtain the following theorem.

Theorem 2.4. *For each nonnegative integer n, we have*

$$\begin{aligned}B_{n+1,\lambda}^{L}(x)=&x\sum_{m=0}^{n}\binom{n}{m}<2>_{n-m}B_{m,\lambda}^{L}(x-\lambda)\\ =&x\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-\lambda)^l l!<l+2>_{k-l}B_{n-k,\lambda}^{L}(x).\end{aligned}\tag{17}$$

By (10) and (11), we know that

$$\sum_{n=0}^{\infty}\mathcal{B}_{n+1,\lambda}(x)\frac{t^n}{n!}=-x\left(1+\lambda\frac{t}{1-t}\right)^{-\frac{x}{\lambda}}\left(1+\lambda\frac{t}{1-t}\right)^{-1}(1-t)^{-2}.\tag{18}$$

Theorem 2.5. *For each nonnegative integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}B_{n-l,\lambda}^{L}(x)\mathcal{B}_{l+1,\lambda}(x)=-x\sum_{m=0}^{n}\binom{n}{m}(-\lambda)^m m!<m+2>_{n-m}.$$

In particular,

$$\sum_{l=0}^{n}\binom{n}{l}B_{n-l+1,\lambda}^{L}\mathcal{B}_{l,\lambda}=-\sum_{m=0}^{n}\binom{n}{m}(-\lambda)^m m!<m+2>_{n-m}.$$

Proof. By multiplying $\left(1+\lambda\frac{t}{1-t}\right)^{\frac{x}{\lambda}}$ on both sides of (18), we get

$$-x\left(1+\lambda\frac{t}{1-t}\right)^{-1}(1-t)^{-2}=\left(\sum_{n=0}^{\infty}\mathcal{B}_{n+1,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B^L_{n,\lambda}(x)\frac{t^n}{n!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B^L_{n-m,\lambda}(x)\mathcal{B}_{m+1,\lambda}(x)\right)\frac{t^n}{n!}.$$

By (9), we get

$$\sum_{m=0}^{n}\binom{n}{m}B^L_{n-m,\lambda}(x)\mathcal{B}_{m+1,\lambda}(x)=-x\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l<1>_l<l+2>_{n-l}.$$

□

If we multiple multiplying $\left(1+\lambda\frac{t}{1-t}\right)^{\frac{x}{\lambda}+1}$ on both sides of (18), then

$$\begin{aligned}\frac{-x}{(1-t)^2}&=\left(\sum_{n=0}^{\infty}\mathcal{B}_{n+1,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B^L_{n,\lambda}(x+\lambda)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n-m+1,\lambda}(x)B^L_{m,\lambda}(x+\lambda)\right)\frac{t^n}{n!}.\end{aligned}\tag{19}$$

By (7) and (19), we have

$$\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n-m+1,\lambda}(x)B^L_{m,\lambda}(x-\lambda)=-x<2>_n.\tag{20}$$

By (20), we obtain the following theorem.

Theorem 2.6. *For each nonnegative integer n, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n-m+1,\lambda}(x)B^L_{m,\lambda}(x+\lambda)=-x<2>_n.$$

In particular,

$$-\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n-m+1,\lambda}B^L_{m,\lambda}(1+\lambda)=<2>_n.$$

3. Conclusion

In the past decades, various degenerate versions of some special polynomials and numbers have been studied actively by many researchers. In particular, degenerate Lah-Bell polynomials were defined by Kim-Kim as degenerate version of Lah-Bell polynomials in [4].

In [14], authors investigated some properties for reciprocal degenerate Bell numbers and polynomials, including their explicit expressions, recurrence relations and their connections with the degenerate Bell numbers and polynomials.

In this paper, we consider reciprocal degenerate Lah-Bell polynomials and numbers and found some properties for those polynomials and numbers including their explicit expressions. In addition, we studied some relationships between the reciprocal Lah-Bell numbers and special numbers and polynomials.

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[1] Department of Mechanical System Engineering, Dongguk University, 123 Dongdae-ro, Gyungju-si, Gyeongsangbuk-do, 38066, Republic of Korea.
Email address: `kbm713@dongguk.ac.kr`

[2] Department of Mathematics, Kyungpook National University, Daegu 41566, Republic of Korea.
Email address: `kimholzi@gmail.com`

[3] Department of Mathematics Education, Daegu University, 38453, Republic of Korea.
Email address: `a0417001@daegu.ac.kr`